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Extension of Derivations

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Any derivation of a properly infinite von Neumann algebra on a Hilbert space into the algebra of bounded operators on this space is implemented by a bounded operator.

INTRODUCTION

Let R be a von Neumann algebra on a Hilbert space H; then R induces two seminorms on the algebra $B(H)$ of bounded operators on H.

The first seminorm $p_1{}^R$ is simply the distance from an operator x to R', the commutant of R, whereas the second $p_2{}^R$ is defined as the norm of $\mathrm{ad}(x)$ restricted to R or, in precise terms,

$$p_2{}^R(x) = \sup\{\| xr - rx \| \mid r \in R \text{ and } \| r \| \leqslant 1\}.$$

In the previous article [2] we found that the algebras mentioned above do have the property that $p_1{}^R$ is equivalent to $p_2{}^R$.

We want to focus on this property, which we call D_0, and we show that if R is a von Neumann algebra such that $R \otimes \mathbb{C}_{l^2(\mathbb{N})}$ has property D_0, then every derivation of R into $B(H)$ is implemented by an operator in $B(H)$. In cohomological terms this means that $H^1(R, B(H)) = 0$.

If, on the other hand, $H^1(R, B(H)) = 0$ for a von Neumann algebra R, then we prove that R has property D_0.

In Section 5 we use some of the previous results to show that certain cohomology groups of the type $H^1(M, N)$ vanish.

1. NOTATION AND PROPERTY D

In our previous publication [2] we showed that, for instance, properly infinite von Neumann algebras have the following property.

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DEFINITION 1.1. A C^* algebra A on a Hilbert space H has property D_0 if and only if there is a positive real k such that for any bounded operator x on H

$$\| \operatorname{ad}(x) \mid A \| \leqslant 2d(x, A') \leqslant k \| \operatorname{ad}(x) \mid A \|.$$

We are not able to show that this property remains after a countable amplification of the algebra so we have to use the following property in some of our computations.

DEFINITION 1.2. A C^* algebra A on a Hilbert space H has property D if and only if $A \otimes \mathbb{C}_{l^2(\mathbb{N})}$ has property D_0.

It is easy to check that both properties D and D_0 survive induction.

As mentioned above, in [2] we show that certain algebras have property D; the results can be found in [2, Sect. 2] and we just list them here.

A von Neumann algebra R has property D if R is properly infinite or R has property P (includes type I) or R is II_1 and generated by commuting sub-von Neumann algebras M and N, which satisfies: M is of type II_1 and has property P.

The notation follows that of [2], but here it is extended by the following cohomological expressions. Let $N \subseteq M$ be von Neumann algebras.

$$Z^1(N, M) = \{\text{the space of derivations from } N \text{ into } M\}.$$
$$B^1(N, M) = \{\operatorname{ad}(m): N \to M \mid m \in M\}.$$
$$H^1(N, M) = Z^1(N, M)/B^1(N, M).$$

We have also included the following notation. If A is an algebra of operators on a Hilbert space H and n is any natural number, then M_n denotes the algebra of complex $n \times n$ matrices and $A^{(n)}$ is the algebra on $H \oplus H \oplus \cdots \oplus H$ (n summands) given by

$$A^{(n)} = \left\{ \left. \begin{pmatrix} x & & & 0 \\ & x & & \\ & & \ddots & \\ 0 & & & x \end{pmatrix} \right| x \in A \right\}.$$

Moreover, if T is a not necessarily bounded operator on the Hilbert space H, then $D(T)$ stands for the domain of definition for T, and if T is preclosed, $\bar{T}$ means the closure of T.

2. UNBOUNDED OPERATORS AND PROPERTY D

The basic result in this section is that, if an algebra has property D_0, then any closed densely defined operator which nearly commutes with the algebra is a sum of a bounded operator and a closed densely defined operator affiliated with the commutant.

PROPOSITION 2.1. *Let A be a unital C^* algebra with property D_0 on a Hilbert space H and let T be a closed densely defined operator on H such that $D(T)$ is left invariant by the elements in A.*

If for any a in A, $(aT - Ta)$ is bounded, then there exists a bounded operator x on H such that for any a in A, $[x, a] = \overline{aT - Ta}$.

Proof. Let δ denote the mapping $a \to \overline{aT - Ta}$; then δ is a derivation of A into $B(H)$, and hence, continuous in the norm topology [6].

In the following computations $[T]$ stands for the range projection of T, s is a positive real, and $V(T^*T)^{1/2}$ is the polar decomposition of T.

Moreover, P_s denotes the projection from $H \oplus H$ onto the graph $G(sT)$ for the operator sT. K_s is defined as $(I + s^2T^*T)^{-1}$.

The computations given in [5, Sect. 118, p. 306] show that P_s, written as a 2×2 matrix with entries from $B(H)$, has the form

$$\begin{pmatrix} K_s & s\overline{K_sT^*} \\ sTK_s & s^2T\overline{K_sT^*} \end{pmatrix}.$$

One should notice that the ranges of K_s and $\overline{K_sT^*}$ are contained in $D(T^*T)$ and $D(T)$, respectively. Remark also that $\overline{T \mid D(T^*T)} = T$, so $\overline{K_sT^*T} = K_sT^*T \mid D(T^*T)$ and $\overline{K_sT^*T} \mid D(T) = \overline{K_sT^*T}$. $(I - P_s)$ is written as

$$\begin{pmatrix} s^2T^*TK_s & -s\overline{K_sT^*} \\ -sTK_s & (I - [T]) + VK_sV^* \end{pmatrix}$$

but for our purpose we prefer the form,

$$\begin{pmatrix} s^2\overline{K_sT^*T} & -s\overline{K_sT^*} \\ -s\overline{VK_sV^*T} & (I - [T]) + VK_sV^* \end{pmatrix}.$$

Let $X = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ be an element in $A^{(2)}$; then a simple computation shows that $(I - P_s)\,XP_s$ as a 2×2 matrix is given by the following four expressions:

$$\begin{aligned} ((I - P_s)\,XP_s)_{11} &= s\overline{K_sT^*}\,(sTa - saT)K_s, \\ ((I - P_s)\,XP_s)_{12} &= s\overline{K_sT^*}\,(sTa - saT)\,s\overline{K_sT^*}, \\ ((I - P_s)\,XP_s)_{21} &= VK_sV^*(saT - sTa)K_s \\ &\quad + (I - [T])\,(saT - sTa)K_s, \\ ((I - P_s)\,XP_s)_{22} &= VK_sV^*(saT - sTa)\,s\overline{K_sT^*} \\ &\quad + (I - [T])\,(saT - sTa)\,s\overline{K_sT^*}. \end{aligned}$$

These equations show that for any pair i, j,

$$\|(I - P_s) XP_s)_{ij}\| \leqslant 2s \|\delta(a)\| \leqslant 2s \|\delta\| \|X\|,$$

and hence,

$$\|\operatorname{ad}(P_s) \mid A^{(2)}\| = \sup\{\|(I - P_s) XP_s\| \mid X \in (A^{(2)})_1\} \leqslant 16s \|\delta\|.$$

On the other hand, for any a in A and $X = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ in $A^{(2)}$ we have

$$(P_s X - XP_s)_{21} = s(TK_s a - aTK_s),$$

and since A has property D_0, there is a positive real r such that for any s in $\mathbb{R}^+$

$$d(sTK_s, A') \leqslant r(16s \|\delta\|).$$

This implies that for any s in $\mathbb{R}^+$ we can find x_s in $B(H)$ such that $\|x_s\| \leqslant 16r \|\delta\|$ and $x_s + TK_s \in A'$.

Let s decrease to zero and take a weakly convergent subnet x_{s_α} with weak limit x; then for any set (ξ, η, u), where ξ is a vector in $D(T)$, η is a vector in H, and u is a unitary in A,

$$\begin{aligned}((T + x)\, \xi \mid \eta) &= \lim_{s_\alpha}((T(I + s_\alpha{}^2 T^*T)^{-1} + x_{s_\alpha})\, \xi \mid \eta) \\ &= \lim_{s_\alpha}((T(I + s_\alpha{}^2 T^*T)^{-1} + x_{s_\alpha})\, u\xi \mid u\eta) = ((T + x)\, u\xi \mid u\eta).\end{aligned}$$

Thus, we have proved that for any unitary u in A and any vector ξ in $D(T)$

$$\delta(u)\xi = (uT - Tu)\xi = (xu - ux)\xi$$

and the proposition follows.

LEMMA 2.2. *Let R be a von Neumann algebra on a Hilbert space H and let $\delta \in Z^1(R, B(H))$. If R has property D_0, then there is a projection p in R' such that*

$$R'p = \{x \in R' \mid x\delta \in B^1(R, B(H))\}.$$

Proof. The proof consists of two steps, the first of which shows that the left ideal N in R' defined by $N = \{x \in R' \mid x\delta \in B^1(R, B(H))\}$ has the property that for any increasing net $(p_\alpha)_{\alpha \in J}$ of projections from N the strong limit will be in N.

The second follows from the first and states that if x belongs to N, so does the support projection of x.

Let $(p_\alpha)_{\alpha \in J}$ be as described with least upper bound p; then since R has property D_0 there is a positive real k such that we for any α in J can find a z_α in $B(H)$ which satisfies

$$\operatorname{ad}(z_\alpha) \mid R = p_\alpha \delta \qquad \text{and} \qquad \|z_\alpha\| \leqslant k \|\delta\|.$$

Take a weakly convergent subnet z_{α_β} with weak limit z; then for each r in R

$$[z, r] = \text{weak lim}[z_{\alpha_\beta}, r] = \text{weak lim}\, p_{\alpha_\beta}\delta(r) = p\delta(r).$$

Therefore, p is in N and our first step is proved; the second step is standard and follows immediately when one considers the spectral projections of x^*x for an element x in N.

COROLLARY 2.3. *Suppose R has property D_0, δ is Hermitian ($\delta(x^*) = \delta(x)^*$), and p is the projection in R' such that $R'p = \{x \in R' \mid x\delta \in B^1(R, B(H))\}$; then for any two projections q and s in R'*

$$\delta - q\delta s \in B^1(R, B(H)) \Leftrightarrow q \geqslant (I - p) \qquad \textit{and} \qquad s \geqslant (I - p).$$

Proof. Suppose $\delta - q\delta s \in B^1(R, B(H))$; then multiplication from the left with $(I - q)$ shows $(I - q) \leqslant p$ and, analogously, $(I - s) \leqslant p$.
If $q \geqslant (I - p)$ and $s \geqslant (I - p)$, then

$$\delta - q\delta s = (I - q)\, \delta s + \delta(I - s) \in B^1(R, B(H)).$$

The results in Proposition 2.1, Lemma 2.2, and Corollary 2.3 are important steps when you want to prove that propeıty D of R implies $H^1(R, B(H)) = 0$, but before we go into these proofs we want to close this section with a partial converse.

THEOREM 2.4. *Let R be a von Neumann algebra on a Hilbert space H and suppose $H^1(R, B(H)) = 0$; then R has proper y D_0.*

Proof. The proof is based upon a closed graph argument, which in this context can be found in [4].

The vector space $B(H)/R'$ is, when equipped with the norm $|||\, x + R'\, ||| = \|\, \mathrm{ad}(x) \mid R\, \|$, isometrically isomorphic to $B^1(R, B(H))$ considered as a subspace of the Banach space $Z^1(R, B(H))$. The condition $H^1(R, B(H)) = 0$ then implies that $B(H)/R'$ with the norm $|||\ \ |||$ is a Banach space, so the closed graph theorem applied to the identity mapping shows that this norm is equivalent to the usual quotient norm on $B(H)/R'$, but this in turn is equivalent to property D_0.

3. THE PROPERLY INFINITE CASE

Let R be a properly infinite von Neumann algebra on a Hilbert space H and let δ be a derivation of R into $B(H)$. Then our procedure is as follows. First, we assume that R has a cyclic vector, say ξ, and second, we show that to such a ξ there exists a vector η such that the operator T' defined by $D(T') = \{r\xi \mid r \in R\}$

and $T'(r\xi) = \delta(r)\xi + r\eta$ is preclosed and densely defined and implements δ. The result then follows from Proposition 2.1 and some elementary technique.

A first important step in the procedure sketched above is the following lemma.

LEMMA 3.1. *Let R be a properly infinite von Neumann algebra on a Hilbert space H and let δ be a derivation of R into $B(H)$, then δ is ultrastrongly-ultrastrongly continuous.*

Proof. Let I_∞ be a sub-von Neumann algebra of R such that I_∞ is isomorphic to all the operators on a separable infinite-dimensional Hilbert space.

From [6, Theorem 7.4, pp. 420–421] we find that there exists an operator x in $B(H)$ such that the kernel of the derivation δ_1 on R defined by $\delta_1 = \delta - \mathrm{ad}(x) \mid R$ contains I_∞.

An argument similar to the one Ringrose gives in [8, Sect. 2] shows that δ_1 is ultrastrongly continuous if and only if there exists a positive real k such that

$$\left\| \sum_{j=1}^{n} \delta_1(r_j)^* \, \delta_1(r_j) \right\| \leqslant k \left\| \sum_{j=1}^{n} r_j^* r_j \right\|$$

for every finite set $r_1 \cdots r_n$ of elements of R [7, Theorem 3.1].

Since I_∞ is infinite, there exists a sequence v_i of isometries in I_∞ such that $(v_i v_i^*)_{i \in \mathbb{N}}$ is a sequence of pairwise orthogonal projections.

Let $(r_1, \ldots, r_n)$ be a finite set of elements from R; then

$$\begin{aligned} \sum_{j=1}^{n} \delta_1(r_j)^* \, \delta_1(r_j) &= \sum_{j=1}^{n} \delta_1(r_j)^* \, v_j^* v_j \, \delta_1(r_j) \\ &= \left(\sum_{j=1}^{n} \delta_1(r_j)^* v_j^* \right) \left(\sum_{i=1}^{n} v_i \delta_1(r_i) \right) \\ &= \left(\sum_{j=1}^{n} \delta_1(v_j r_j) \right)^* \left(\sum_{i=1}^{n} \delta_1(v_i r_i) \right), \end{aligned}$$

and hence,

$$\begin{aligned} \left\| \sum_{j=1}^{n} \delta_1(r_j)^* \, \delta_1(r_j) \right\| &= \left\| \sum_{i=1}^{n} \delta_1(v_i r_i) \right\|^2 \leqslant \| \delta_1 \|^2 \left\| \sum_{i=1}^{n} v_i r_i \right\|^2 \\ &= \| \delta_1 \|^2 \left\| \sum_{i=1}^{n} r_i^* r_i \right\|. \end{aligned}$$

We now follow Ringrose [8] and show that this condition implies that to any positive normal functional φ on $B(H)$ there is a positive functional ω on R such that

$$\forall r \in R \colon \varphi(\delta_1(r)^* \, \delta_1(r)) \leqslant \omega(r^* r). \tag{1}$$

Let $S_1 = \{r \in R \mid \varphi(\delta_1(r)^* \delta_1(r)) = 1\}$, let S_2 be the convex hull of the set $\{r^*r \mid r \in S_1\}$, and $x = \sum_{i=1}^{n} \lambda_i r_i{}^* r_i$ be an element of S_2; then

$$\| x \| = \left\| \sum_{i=1}^{n} (\lambda_i^{\frac{1}{2}} r_i)^* (\lambda_i^{\frac{1}{2}} r_i) \right\| \geqslant \| \delta_1 \|^{-2} \left\| \sum \lambda_i \delta_1(r_i)^* \delta_1(r_i) \right\|$$

$$\geqslant \| \delta_1 \|^{-2} \| \varphi \|^{-1}.$$

The Hahn–Banach theorem then shows that there will be a Hermitian functional ψ on R such that $\psi(x) \geqslant 1$ whenever X is in S_2. If we let ω be the positive part of ψ, then

$$\forall r \in R\colon \varphi(\delta_1(r)^* \delta_1(r)) \leqslant \omega(r^*r). \tag{2}$$

In the proof in [8] Ringrose uses his Lemma 2.2 to show that (2) will also be fulfilled with a normal functional instead of ω. Here, we want to show directly that the normal part ω_{n} of ω will do.

Recall from [10] that every functional on a von Neumann algebra has a unique decomposition as a sum of an ultraweakly continuous functional and a singular functional.

Further remember that, if we let ω_{n} and ω_{S} be the normal and singular parts in this decomposition of ω then the kernel of ω_{S} contains a family $(q_\alpha)_{\alpha \in]}$ of pairwise orthogonal projections from R with sum I. This family, as one easily finds, induces a, toward I, strongly convergent net $(p_\beta)_{\beta \in D}$ of projections from the kernel of ω_{S}.

Now let $r \in R$; then since δ_1 is ultraweakly continuous [7],

$$\begin{aligned}
\varphi(\delta_1(r)^* \delta_1(r))^2 &= \lim_{\beta} | \varphi(\delta_1(r)^* \delta_1(rp_\beta))|^2 \\
&\leqslant \lim\sup \varphi(\delta_1(r)^* \delta_1(r))\, \varphi(\delta_1(rp_\beta)^* \delta_1(rp_\beta)) \\
&\leqslant \varphi(\delta_1(r)^* \delta_1(r)) \lim\sup \omega(p_\beta r^* rp_\beta) \\
&= \varphi(\delta_1(r)^* \delta_1(r)) \lim \omega_n(p_\beta r^* rp_\beta) = \varphi(\delta_1(r)^* \delta_1(r))\, \omega_n(r^*r)
\end{aligned} \tag{3}$$

Finally, we obtain

$$\forall r \in R\colon \varphi(\delta_1(r)^* \delta_1(r)) \leqslant \omega_n(r^*r) \tag{4}$$

and we find that δ_1 is ultrastrongly continuous.

THEOREM 3.2. *If R is a properly infinite von Neumann algebra on a Hilbert space H, then $H^1(R, B(H)) = 0$.*

Proof. We have to show that $Z^1(R, B(H)) = B^1(R, B(H))$, but since $Z^1(R \cap R', B(H)) = B^1(R \cap R', B(H))$, we need only prove that any Hermitian element δ of $Z^1(R, B(H))$ which is trivial on $R \cap R'$ belongs to $B^1(R, B(H))$. Suppose $\delta \in Z^1(R, B(H)) \backslash B^1(R, B(H))$; then since R has property D_0 there is a projection q in R' such that for any projection s in R' smaller than q but not zero, we have that $\delta - q\delta q$ is implemented and $s\,\delta q$ is not.

We have assumed that δ is trivial on the center of R; therefore, if r is in R and $rq = 0$ we get that $r\,c(q) = 0$ (where $c(q)$ denotes the central support of q), and

$$q\,\delta(r)\,q = q\,c(q)\,\delta(r)\,q = q\,\delta(c(q)\,r)\,q = 0.$$

This last line implies that we may restrict our attention to a derivation δ_1 of R_q into $B(qH)$ defined by $\delta_1(rq) = q\,\delta(r)\,q$.

More generally, we assume that we have a derivation δ of the properly infinite von Neumann algebra R into $B(H)$ which has the properties that it is Hermitian, trivial on the center, and $\{x \in R' \mid x\delta \in B^1(R, B(H))\} = \{0\}$.

Let $\delta^{(\infty)}$ be the derivation on $R^{(\infty)}$ given by

$$\delta^{(\infty)}\left(\begin{pmatrix} \ddots & & & 0 \\ & r & & \\ & & r & \\ 0 & & & \ddots \end{pmatrix}\right) = \begin{pmatrix} \delta(r) & & & \\ & \delta(r) & 0 & \\ & 0 & \ddots & \end{pmatrix}$$

and suppose that $x \in R' \otimes B(l^2(\mathbb{N}))$ is chosen such that there is a z in $B(H) \otimes B(l^2(\mathbb{N}))$ for which $x\delta^\infty = \mathrm{ad}(z) \mid R^{(\infty)}$.

For any natural numbers i, j we then get, when the tensor products are written as infinite matrices, that the entry x_{ij} belongs to R' and for any r in R, $x_{ij}\,\delta(r) = z_{ij}r - r\,z_{ij}$. We can then conclude that x equals zero and that we, in the following arguments, may suppose that R' is also properly infinite.

We first give the proof in the special case where R has a cyclic vector say ξ. Let us then recall that δ is ultrastrongly continuous, from which the homomorphism Φ of R into $B(H) \otimes M_2$, given by

$$\Phi(r) = \begin{pmatrix} r & 0 \\ \delta(r) & r \end{pmatrix}$$

is ultrastrongly continuous.

This, in turn, implies that there exists a positive normal functional ψ on R such that for any r in R

$$\| r\xi \|^2 + \| \delta(r)\xi \|^2 = \left\| \begin{pmatrix} r & 0 \\ \delta(r) & r \end{pmatrix} \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\|^2 \leqslant \psi(r^*r).$$

Since R' is properly infinite there exists a vector η in H such that for any r in R

$$\psi(r) = (r\eta \mid \eta),$$

and hence,

$$\| r\xi \|^2 + \| \delta(r)\,\xi \|^2 \leqslant \| r\eta \|^2.$$

Let C denote the image of the homomorphism Φ and define subspaces $H_1 = H \oplus \{0\}$, $H_2 = \{0\} \oplus H$ of $H \oplus H$, and projections e_1, e_2 of $H \oplus H$ onto H_1 and H_2, respectively. Define the closed invariant subspace L for the algebra C as the closure of $\{\left(\begin{smallmatrix} r & r \\ \delta(r) & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \xi \\ 2\eta \end{smallmatrix}\right) \mid r \in R\}$, and let q denote the projection from H_2 onto $H_2 \cap L$.

It is easy to see that $q \in R'$ and not too difficult to find that $\left(\begin{smallmatrix} I & 0 \\ 0 & I-q \end{smallmatrix}\right) L$ is the graph of a closed densely defined operator T on H. T is given in the following manner: $D(T) = e_1 L$ and for $\gamma \in D(T)$, $T\gamma$ is defined by $(\gamma, T\gamma) \in L$, and $(I - q)\, T\gamma = T\gamma$ (in the arguments below one needs that if $(x, y) \in L$ then $x \in D(T)$ and $Tx = (I - q)\, y$).

T is automatically closed and T is densely defined because $D(T) = e_1 L = \{r\xi \mid r \in R\}$, and ξ is cyclic.

The special choice of η is made in order to ensure that q does not equal the identity. Suppose $q = I$; then we find that $L = H \oplus H$, but for any r in R, $\Phi(r)(\xi, 2\eta) = (r\xi, \delta(r)\, \xi + 2r\eta)$, and $\| \delta(r)\, \xi + 2r\eta \| \geqslant \| r\eta \| \geqslant \| r\xi \|$. Hence, for any (x, y) in L, $\| x \| \leqslant \| y \|$, which contradicts $L = H \oplus H$.

If we define the algebra D on $H \oplus H$ by

$$D = \left\{ \begin{pmatrix} r & 0 \\ (I - q)\, \delta(r) & r \end{pmatrix} \middle| \, r \in R \right\},$$

then the graph $G(T)$ of T is an invariant subspace for D; therefore, for any γ in $D(T)$ and any r in R, $r\gamma \in D(T)$ and $Tr\gamma = (I - q)\, \delta(r)\, \gamma + rT\gamma$.

Application of Lemma 2.1 yields that $(I - q)\, \delta \in B^1(R, B(H))$ and we have a contradiction.

We now return to the case without a cyclic vector, but still with the other assumptions on R and δ.

Let P be the set of projections in R' which is given by $p \in P$ if and only if p is σ-finite with respect to R' and $R_p{}'$ is properly infinite. Standard techniques show that if p and q are elements of P then $p \vee q \in P$ and if s is any σ-finite projection in R', then there is a projection p in P such that $s \leqslant p$.

Take an element p in P; then $R_p{}'$ has a faithful normal state, and in fact, this state can be represented as a vector state ω_ξ because R_p is properly infinite. The vector ξ is then cyclic for R_p and we find that the derivation δ_p of R_p into $B(pH)$ given by $\delta_p(rp) = p\, \delta(r)\, p$ is implemented by an operator x_p on pH with norm less than $2 \| \delta \|$.

If we let the operators x_p act on all of H canonically, we find that for any p in P and any r in R,

$$p\delta(r)\, p = [x_p, r].$$

Since $(p)_{p \in P}$ converges strongly to I a compactness argument gives the desired contradiction.

4. The Finite Case

The main reason that we cannot just incorporate the result we get in the finite case in the result for the properly infinite case is that we are not able to prove automatic ultrastrong continuity for derivations of a finite von Neumann algebra. On the contrary, we do get rather easily that if M is an arbitrary finite von Neumann algebra and N is a unital C^* algebra contained in M, then $H^1(N, M) = 0$.

Lemma 4.1. *Suppose M is a finite von Neumann algebra on a Hilbert space H, and that M has a separating and cyclic trace vector ξ.*

If M has property D_0, then $H^1(M, B(H)) = 0$.

Proof. The idea is to show that if $\delta \in Z^1(M, B(H))$, then the operator T_0 on H given by $D(T_0) = M\xi$ and $T_0(m\xi) = \delta(m)\xi$ is densely defined and preclosed.

Since ξ is separating and cyclic, T_0 is well defined and densely defined; suppose that $(m_k)_{k\in\mathbb{N}}$ is a sequence of operators in M such that $m_k\xi$ converges in norm to zero and $\delta(m_k)\,\xi = T_0(m_k\xi)$ converges in norm to a vector γ different from zero.

Let tr denote the trace $m \to (m\xi \mid \xi)$ on M; then the theory of noncommutative integration [9] tells that tr can be extended to a certain subspace of the closed densely defined operators affiliated with M. This subspace is called the space of tr integrable operators and it has the properties that if m is in M and S is tr integrable, then mS is tr integrable. Moreover, to any ultraweakly continuous functional φ on M there is a unique integrable operator S such that for any m in M; $\varphi(m) = \mathrm{tr}(Sm)$.

From [7] it is seen that δ is ultraweakly continuous; therefore, there exists a tr integrable operator K such that for any m in M, $(\delta(m)\,\xi \mid \xi) = \mathrm{tr}(mK)$.

Let $i \in \mathbb{N}$ and let e_i be the spectral projection for $(K^*K)^{1/2}$ corresponding to the interval $[0, i]$ of $\mathbb{R}$; then for fixed i and any b in M, Ke_i is bounded and

$$\lim_{k\to\infty} \delta(e_i b m_k)\,\xi = \lim_{k\to\infty}(\delta(e_i b)\, m_k\xi) + \lim_{k\to\infty}(e_i b\delta(m_k)\,\xi) = e_i b\gamma.$$

Hence,

$$\begin{aligned}(\gamma \mid b^* e_i\xi) &= (e_i b\gamma \mid \xi) = \lim_{k\to\infty}(\delta(e_i b m_k)\,\xi \mid \xi) = \lim_{k\to\infty} \mathrm{tr}(e_i b m_k K)\\ &= \lim_{k\to\infty} \mathrm{tr}((Ke_i)\, b m_k) = \lim_{k\to\infty}(m_k\xi \mid b^*(Ke_i)^*\xi) = 0.\end{aligned}$$

Therefore, if we let $i \to \infty$, we obtain $(\gamma \mid b^*\xi) = 0$ for any b in M, and finally, $\gamma = 0$ so T_0 is preclosed.

Let T denote the closure of T_0 and let $\eta \in D(T)$; then there exists a sequence of operators m_k in M such that $m_k\xi$ converges to η and $\delta(m_k)\,\xi$ converges to $T\eta$.

A simple computation shows that for any b in M, $bm_k\xi$ converges to $b\eta$ and $\delta(bm_k)\,\xi$ converges to $bT\eta + \delta(b)\,\eta$, which implies that $b\eta \in D(T)$ and $Tb\eta = bT\eta + \delta(b)\,\eta$. Lemma 2.1 then yields the conclusion.

THEOREM 4.2. *Let M be a finite von Neumann algebra on a Hilbert space H. If M has property D, then $H^1(M, B(H)) = 0$.*

Proof. Let δ be a derivation of M into $B(H)$; then we may assume that δ is trivial on the center and restrict our attention to the case where M has a bounded faithful normal trace.

We let $\mathscr{M}$ denote the standard representation of M; then there are index sets I and J such that

$$M \otimes \mathbb{C}_{(l^2(I))} \qquad \text{and} \qquad \mathscr{M} \otimes \mathbb{C}_{l^2(J)}$$

are spatially isomorphic. Let us then find matrix units $(e_{kl})_{k,l\in J}$ in $M' \oplus B(l^2(I))$ such that the $(M \otimes \mathbb{C}_{(l^2(I))})_{e_{kk}}$ are spatially isomorphic to $\mathscr{M}$.

For any fixed element i_0 of J and any arbitrary elements k, l of J, we define a derivation $\delta_{k,l}$ of $(M \otimes \mathbb{C}_{l^2(I)})_{e_{i_0i_0}}$ into $B(e_{i_0i_0}(H \otimes l^2(I))$ by

$$\delta_{kl}((xe_{i_0i_0})) = e_{i_0k}\delta(x)e_{li_0}\,.$$

Lemma 4.1 and the assumptions and constructions above show that there is an operator v_{kl} on $e_{i_0i_0}(H \otimes l^2(I))$, which implements δ_{kl}.

Define $w_{kl} = e_{ki_0i}v_{kl}e_{i_0l}$ and define for any finite subset J_0 of J, $E_{J_0} = \sum_{j\in J_0} e_{jj}$; then for any x in $M \otimes \mathbb{C}_{(l^2(I))}$ we get

$$\begin{aligned} E_{J_0}\delta(x)E_{J_0} &= \sum_{k,l\in J_0} e_{ki_0}(e_{i_0k}\delta(x)e_{li_0})e_{i_0l} \\ &= \sum_{k,l\in J_0} [w_{kl}\,,\, x] = \Big[\sum_{k,l\in J_0} w_{kl}\,,\, x\Big]. \end{aligned}$$

Since $M \otimes \mathbb{C}_{(l^2(I)}$ has property D, there exists a positive real K such that for any finite subset J_0 of J there is an operator z_{J_0} for which

$$\| z_{J_0} \| \leqslant K \| \delta \| \qquad \text{and} \qquad E_{J_0}\delta(x)E_{J_0} = [z_{J_0}\,,\, x].$$

As usual, a compactness argument gives the final condusion.

5. COHOMOLOGY

We start this section with the proof of the result mentioned in Section 4, namely, that $H^1(M, N) = 0$ if M is contained in N and N is finite.

The theorem below is formulated in a more general fashion, such that it will also cover the case where N is the crossed product of a finite algebra N_0 by a single automorphism and M is contained in N_0.

THEOREM 5.1. *Suppose that $M \subseteq N_0 \subseteq N$ are von Neumann algebras, that N_0 is finite, and that π is a faithful normal projection of norm 1 from N onto N_0; then $H^1(M, N) = 0$.*

Proof. Let $\delta \in H^1(M, N)$.

As in the case where $M = N$ [6, p. 382], we study the weakly compact, closed convex hull K of the set $\{\delta(u^*)\, u \mid u \text{ unitary in } M\}$ and the group $\{A_u \mid u \text{ unitary in } M\}$ of affine transformations on K given by

$$A_u(k) = u^*ku + u^*\delta(u).$$

Let τ be any finite trace on M and let k_1 and k_2 be elements in K; then for any unitary u in M, $A_u(k_1) - A_u(k_2) = u^*(k_1 - k_2)\, u$. Therefore,

$$\begin{aligned}\tau(\pi((A_u(k_1) &- A_u(k_2))^* (A_u(k_1) - A_u(k_2))))\\ &= \tau(u^*\pi((k_1 - k_2)^* (k_1 - k_2))\, u)\\ &= \tau(\pi((k_1 - k_2)^* (k_1 - k_2))).\end{aligned}$$

This shows that the group A_u is noncontracting and the Ryll–Nardzewski fixed point theorem yields the final conclusion.

The following lemma is more or less trivial, but it helps us to get some results which are not quite obvious.

LEMMA 5.2. *Let $A \subseteq B \subseteq C$ be von Neumann algebras on a Hilbert space H; then there is a canonical injective homomorphism φ of*

$$Z^1(A, B) \cap B^1(A, C)/B^1(A, B)$$

into

$$Z^1(B', A') \cap B^1(C', A')/B^1(B', A').$$

Proof. Take c in C such that $\mathrm{ad}(c)$ maps A into B and look at $\mathrm{ad}(c) \mid B'$. An easy computation shows that $\mathrm{ad}(c)$ maps B' into A' and that for any b in B $\mathrm{ad}(c + b) \mid B' = \mathrm{ad}(c) \mid B'$; hence, it follows that we have a well-defined map φ between the two sets mentioned in the lemma when we define

$$\varphi(\mathrm{ad}(c) + B^1(A, B)) = \mathrm{ad}(c) + B^1(B', A').$$

If $\mathrm{ad}(c)$ restricted to B' is of the form $\mathrm{ad}(a') \mid B'$ for an a' in A', then $c - a'$ belongs to B, and hence, $\mathrm{ad}(c) \in B^1(A, B)$. The lemma follows.

THEOREM 5.3. *Let A be a von Neumann algebra on a Hilbert space H.*
If $H^1(A, B(H)) = 0$ and A' is finite, then $H^1(A, B) = 0$ for any von Neumann algebra B on H containing A.

Proof. Consider $A \subseteq B \subseteq B(H)$; then the assumptions on A show that $H^1(B', A') = 0$ and $H^1(A, B(H)) = 0$, so Lemma 5.2 implies that $H^1(A, B) = 0$.

THEOREM 5.4. *Suppose $A \subseteq B \subseteq C$ are von Neumann algebras if $H^1(A, C) = 0$ and A or B has the extension property; then $H^1(A, B) = 0$.*

Proof. Suppose first that B has the extension property.
Take δ in $Z^1(A, B)$, find c in C such that $\delta(a) = [c, a]$, and let π be a projection norm onto B.
It is then obvious that δ is implemented by $\pi(c)$ and $\delta \in B^1(A, B)$.
Let us now assume that A has the extension property and that δ and c are as above. We find that $\mathrm{ad}(c)$ maps B' into A' and hence, as above, that for any projection ρ of norm 1 onto A'; $(c - \rho(c)) \in B$. The theorem follows.

COROLLARY 5.5. *If A or B has the extension property*

$$\{c \in C \mid \mathrm{ad}(c): A \to B\} = C \cap \{a' + b \mid a' \in A',\ b \in B\}.$$

Proof. Identical to the proof above.

COROLLARY 5.6. *Suppose $A \subseteq B$ are von Neumann algebras on a Hilbert space H; then*

$$H^1(A, B) = 0$$

if A has property P or A is properly infinite and A or B has the extension property.

Proof. In any of the cases, $H^1(A, B(H)) = 0$, so the theorem applies.

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